

Symmetry as a source of hidden coherent structures in quantum physics: general outlook and examples

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Abstract

A general algebraic approach, incorporating both invariance groups and dynamic symmetry algebras, is developed to reveal hidden coherent structures (closed complexes and configurations) in quantum many-body physics models due to symmetries of their Hamiltonians H . Its general ideas are manifested on some recent new examples: 1) G -invariant bi-photons and a related $SU(2)$ -invariant treatment of unpolarized light; 2) quasi-spin clusters in nonlinear models of quantum optics; 3) construction of composite particles and (para)fields from G -invariant clusters due to internal symmetries.

1 Introduction. General remarks

The symmetry methods are widely used in quantum physics from the time of its origin and up to now because they yield powerful epistemological and computational tools for examining many physical problems (see, e.g., [1-14] and literature cited therein). In particular, invariance principles provide formulations of dynamic laws and classifications of quantum states which are most adequate to reveal different physical phenomena [1, 2] whereas the formalism of groups and Lie algebras, especially, generalized coherent states and related techniques, yield simple and elegant solutions of spectral and evolution problems [9, 13]. From the spectroscopic point of view one distinguishes two (exploiting, as a rule, independently) types of physical symmetries depending on the behaviour of Hamiltonians H under study with respect to symmetry transformations [5]. One of them, associated with invariance groups $G_i(H)$ ($[G_i, H]_- = 0$) of Hamiltonians, describes (non-accidental) degeneracies of energy spectra within fixed irreducible representations (IRs) of $G_i(H)$ while another one, connected with so-called dynamic symmetry (or spectrum generating) algebras g^D ($[g^D, H]_- \subseteq g^D \neq 0$) [5], enables to determine such spectra within fixed IRs of g^D and to give spectral decompositions of Hilbert spaces $L(H)$ of quantum systems in g^D -invariant subspaces $L(\lambda)$ (with λ being labels of g^D IRs D^λ) which describe certain (macroscopic) coherent structures (CS), i.e., stable sets of states (shells, (super)multiplets, configurations, phases, etc. [3-12]) evolving in time independently under actions of H .

Applications of these methods are especially fruitful in examining many-body problems whose Hamiltonians H and quantum state spaces $L(H)$ are given in terms of boson-fermion operators: $H = H(a_i, a_i^+, b_j, b_j^+)$, $L(H) \subseteq L_F(n; m) \equiv \text{Span}\{\prod_{i=1}^n (a_i^+)^{\eta_i} \prod_{j=1}^m (b_j^+)^{\mu_j} |0\rangle\}$ ($[a_i, a_j^+]_- = \delta_{ij} = [b_i, b_j^+]_+$). Indeed, various (originated from the works [15]) boson-fermion mappings f :

$$(a_i, a_i^+, b_j, b_j^+) \xrightarrow{f} F_\alpha, \quad [F_\alpha, F_\beta]_\mp \equiv F_\alpha F_\beta \mp F_\beta F_\alpha = \sum_{\gamma=1}^d c_{\alpha\beta}^\gamma F_\gamma \equiv \psi_{\alpha\beta}(\{F_\alpha\}) \quad (1)$$

enable us to introduce generators F_β of $d (< \infty)$ -dimensional Lie algebras (or superalgebras [11]) as (super)symmetry operators of both types and collective dynamic variables of problems under study in whose terms one gets reformulations of $H, L(H)$ facilitating solutions of many, mainly, spectroscopic many-body tasks [3-12]. On the other hand, within many-body models, due to composite structures of their "elementary" coupled micro-objects (quasi-particles, clusters, etc.), one can reveal in a natural manner deep (although hidden) interrelations between both symmetry types above, and, therefore, a study of one of them automatically yields an information about other one[16]. A consequent realization of this standpoint, being complemented by an "invariant confinement principle" (for constituents), leads to an unified "invariant-dynamic" approach (IDA) to reveal new cooperative effects and phenomena in many-body physics on both micro- and

macro-levels [17]; from the methodological point of view it may be considered as a specification of the general nature-philosophical principle (used at the intuitional level already by I. Kepler and explicitly realized by H. Weyl and E. Wigner within modern physics [1, 2]): *symmetry generates (induces) a formation of CS (coherent configurations) in sets of interacting objects.*

Note that single (mostly, formal) aspects of IDA were implemented in quantum physics long ago, beginning from using binary $SU(2)$ -invariants to describe bi-polar molecular valent bonds in 1931 [18] and from group-theoretical studies of complex atom spectra by G. Racah in 1942-49 [19]. Specifically, the latters were fruitfully developed later in nuclear, atomic and molecular physics [3, 4, 6, 12] and have led to exact definitions of two basic concepts of IDA [13, 20]: dynamic symmetry algebras (see [4, 5] and references therein) and complementarity of groups [21]. The most complete (although implicit) implementations of IDA were given in superfluidity/superconductivity theories by introducing $U(1)$ -invariant Bogolubov's/Cooper's pairs and the $SU(1, 1)/SU(2)$ canonical transformations [22] and in particle physics (interrelations between "colour" and "flavour" $SU(n)$ symmetries) [10]. However, up to recently an explicit mathematical formulation of IDA, summing up such implementations, was absent that prevented its systematic applications. The aim of the paper is to give (destined for physicists) mathematical grounds of IDA [13, 23] and to manifest its efficiency and physical meaning on some recently examined examples in quantum optics [20, 24-27] and in the theory of composite particles and fields with internal symmetries [13, 23].

The paper is dedicated to the memory of Academician N.N. Bogolubov, whose ideas and works promoted to forming IDA, and of Professor Ya. A. Smorodinsky, discussions with whom stimulated the developments presented below.

2 Mathematical grounds: G -invariant Jordan mappings and Weyl-Howe dual pairs in many-body physics

The mathematical formulation of IDA is based on a synthesis of vector invariant theory [1, 29] and extensions [13, 20] of the concept of complementary groups and of the Jordan mapping [15].

As is known, the original Jordan mapping, given by Eq. (1) with quadratic functions f , introduces collective dynamic variables $F_\alpha(t)$ related to generators F_α of certain Lie (super) algebras g_0^D of dynamic symmetry and reduces quadratic (in field operators) Hamiltonians $H_0(a_i, a_i^+, b_j, b_j^+)$, describing free and linear (in the Heisenberg picture) dynamics, to the form

$$H_0 = \sum_{\alpha} \lambda_{\alpha} F_{\alpha} + C, \quad [F_{\alpha}, C]_- = 0, \quad \text{Span}\{F_{\alpha}\} = g_0^D \quad (2)$$

where λ_{α} are c -number coefficients; herewith algebras g_0^D for particular H are subalgebras of certain "maximal" (in a sense) finite-dimensional Lie superalgebras g_0^{DM} which act on $L_F(n; m)$ irreducibly and are semi-direct products of the superalgebras $osp(2n|2m)$ (with the even part $sp(2n, R) \otimes o(2m)$) and the Weyl-Heisenberg superalgebras $w(n, m) = \text{Span}(a_i, a_i^+, b_j, b_j^+)$ [5, 11, 12, 13].

Suppose now that H_0 have (both continuous and discrete) invariance groups $G_i(H_0) = G_i^0$ and field operators form sets of vectors $a_i^+ = (a_{ki}^+), b_j^+ = (b_{kj}^+)$ which are transformed with respect to some (e.g., fundamental) IRs of the groups G_i^0 . Then $F_\alpha \in g_0^D$ are quadratic vector G_i^0 -invariants and, besides, there is the characteristic equation: $[g_0^D, G_i^0]_- = 0 \iff G_i^0 g_0^D G_i^{0-1} = g_0^D$ entailing functional connections between G_i - and g_0^D -invariant (Casimir and class [3]) operators $C_k(G_i)$ and $C_k(g_0^D)$ and specifications of their eigenvalues on spaces $L(H)$ by common sets $[l_i] \equiv [l_0, l_1, \dots]$ of invariant quantum numbers l_i which determine IRs of both G_i and g_0^D and label their common extremal (usually, lowest) vectors $|[l_i]\rangle$ [23]. All that, in turn, yields spectral decompositions

$$L(H) = \sum_{[l_i]} \sigma([l_i]) L([l_i]), \quad L([l_i]) = \text{Span}\{D^{[l_i]}(G_i^0) \otimes D^{[l_i]}(g_0^D)|[l_i]\}\} \quad (3)$$

of spaces $L(H)$ in direct sums of the subspaces $L([l_i])$ which are invariant with respect to joint actions $D(g_0^D) \otimes D(G_i^0)$ of algebras g_0^D and groups G_i^0 being carrier-spaces of so-called isotropic

components (factor-representations) [5] of both these algebraic structures, i.e. $L([l_i])$ contains carrier-spaces of equivalent IRs $D^{[l_i]}(G_i^0)$ ($D^{[l_i]}(g_0^D)$) with multiplicities being equal to dimensions of IRs $D^{[l_i]}(g_0^D)$ ($D^{[l_i]}(G_i^0)$). In the case of suitable (for given H and $L(H)$) groups G_i^0 decompositions (3) have the simple spectra: $\sigma([l_i]) = 1$, and, then, pairs (G_i^0, g_0^D) (or (G_i^0, G_0^D) , $G_0^D = \exp(g_0^D)$, $G_i^0 \otimes G_0^D \subset G_0^{DM}$) say to act complementarily [21, 28] on $L(H)$ and to form the Weyl-Howe dual pairs [20] since pairs $(G_i^0 = S_N, G_0^D = U(n))$ of permutation and unitary groups were first considered within quantum mechanics by H. Weyl [1], and their explicit mathematical characterization for pairs $(O(n), Sp(2m, R))$ of orthogonal and symplectic groups was given by R. Howe [29] (from hereon indices $i, 0, D$ are omitted whenever it is of no importance). Note that implicitly such Weyl-Howe dual pairs were used in different fields of many-body physics (see, e.g., [28, 20] and references therein); without dwelling on a review of these applications we mention some of known examples: pairs $(SU(n), SU(m))$ in particle physics [10], $(U(1), su(1, 1) \subset sp(2m, R))$ ($m \rightarrow \infty$) in superfluidity theory [9] and $(C_2, SU(1, 1))$ in describing so-called squeezed light [30].

The constructions above are generalized in a natural manner when extending quadratic Hamiltonians H_0 by G_i^I ($\subseteq G_i^0$)-invariant polynomials $H_I(a_i, a_i^+, b_j, b_j^+)$ of higher degrees which describe essentially nonlinear interactions [13, 31] (and, often, with enlarging Hilbert spaces $L(H)$). In general, such extensions lead to dual pairs where dynamic algebras g^D are infinite-dimensional graded Lie (super)algebras $g^D = \sum_{r=-\infty}^{\infty} g_r$, $[g_r, g_s] \subset g_{r+s}$ enlarging Lie algebras g_0^D and embedded into enveloping algebras $\mathcal{U}(w(n; m))$ of algebras $w(n; m)$ (from hereon we omit the subscript "±" in $[,]_{\pm}$ whenever it is unnecessary) [17]. However G_i -invariance of H enables us to get generalized dual pairs $(G_i^I, g^D = \hat{g})$ where dynamic symmetry is described by finite-dimensional non-linear (polynomial) Lie (super)algebras $\hat{g} = g_0^D + y_+ + y_-$ extending Lie algebras g_0^D and having an independent meaning. These algebras \hat{g} are introduced with the help of G_i^I -invariant polynomial Jordan mappings which in the simplest case, when $H_I(\dots)$ are homogeneous polynomials in a_i, a_i^+, b_j, b_j^+ , has the form \tilde{f} [13, 17]

$$(a_i, a_i^+, b_j, b_j^+) \xrightarrow{\tilde{f}} (F_\alpha, Y_\lambda, Y_\lambda^+) \in \hat{g}, \quad [g_0^D, y] \subseteq y, \quad [y, y] \subset \mathcal{U}(g_0^D), \quad y = y_+ + y_- \quad (4)$$

where generators $Y_\lambda \in y_-, Y_\lambda^+ \in y_+$ are simultaneously elementary vector G_i -invariants [1] and components of two mutually conjugate g_0^D -irreducible tensor operators Y, Y^+ . In practice, Hamiltonians H_0, H_I may be inhomogeneous polynomials in a_i, a_i^+, b_j, b_j^+ and, besides, contain other g_0^D -covariant operators that leads to modifications of Eq. (4) [20, 26]. The first example of using the mapping (4) in physical problems was given (implicitly) in [31] for extending the unitary algebra $u(1)$ by its C_n -invariant symmetric tensors; later such constructions were introduced explicitly in [17, 20] for extending algebras $u(m)$ by their C_n -invariant symmetric and $SU(n)$ -invariant skew-symmetric tensor operators (see Section 5) as well as for extending the symplectic algebras $sp(2m, R)$ by $SO(n)$ -invariant skew-symmetric tensors.

Without dwelling on a complete analysis of the algebras \hat{g} we outline some of their features. As is seen from Eq. (4), algebras \hat{g} resemble in their structure so-called q -deformed Lie algebras (widely used for last time [14]) and have the coset structure (generalizing the Cartan decomposition for real semisimple algebras [5]) that enables us to construct IRs of \hat{g} starting from g_0^D -modules. However, unlike usual (linear) Lie algebras, exponentials $\exp(\hat{g})$ generate only pseudogroup structures rather than finite-dimensional Lie groups (cf. [32]) that impedes direct extensions of standard group-theoretical techniques for solving physical tasks [23]. Nevertheless, using generalizations [20]

$$(F_\alpha, Y_\lambda, Y_\lambda^+) \xrightarrow{\tilde{f}} (F_\alpha^0 = F_\alpha, F_\alpha'^+, F_\alpha') \in h, \quad F_\alpha'^+ = (F_\alpha')^+ = \sum_{\lambda} Y_\lambda^+ f_{\alpha, \lambda}(\{F_\beta\}), \quad [h, h] \subseteq h \quad (5)$$

of the Holstein-Primakoff mappings [15] (with h being usual Lie (super)algebras and "coefficients" $f_{\alpha, \beta}(\dots)$ determined from sets of finite-difference equations), one can construct some finite-dimensional Lie subgroups $\exp(h) \subset \exp(\hat{g})$ which are useful for physical applications [23].

Let us now sketch some of physical aspects of formal constructions above to elucidate the heuristic meaning of IDA. The key role belongs here to the decomposition (3) which describes

"kinematic" premises of arising CS in $L(H)$ due to the G_i symmetry. Indeed, subspaces $L([l_i])$ consist of the " g^D -layers" $L([l_i]; \nu) = \text{Span}\{|[l_i]; \mu; \nu\rangle = \mathcal{P}_\nu^{[l_i]; D}(F_\alpha^+, Y_\lambda^+)|[l_i]; \mu\rangle\}$ obtained by actions of polynomials in the g^D positive weight shift generators on basic vectors $|[l_i]; \mu\rangle = \mathcal{P}_\mu^{[l_i]; i}|[l_i]\rangle$ of the IRs $D^{[l_i]}(G_i)$ which are simultaneously specific (degenerated) "pseudovacuum" vectors with respect to $g^D : Y_\lambda|[l_i]; \mu\rangle = F_\alpha|[l_i]; \mu\rangle = 0$. Thus, G_i -invariance plays a "synergetic" role and yields "potential (kinematic) forms" for CS which may be formed in $L(H)$ and are described by subspaces $L([l_i])$ at the macroscopic level and by g^D -cluster variables F_α^+, Y_λ^+ at the microscopic level. Note that, generally, the decompositions (3) contain the "particular" (G_i -scalar) subspaces $L([0])$ "consisting" only of g^D -clusters whereas other spaces $L([l_i])$ "contain" fixed (determined by the "signatures" $[l_i]$) numbers of uncoupled or partially coupled "primary particles". "Physical" realizations of these hidden CS are implemented dynamically in their "pure" or "mixed" kinematic forms determined by concrete G_i -invariant Hamiltonians H_I (containing or not G_i -covariant coupling parameters (fields) "mixing" different $L([l_i])$) and initial states $|\psi(0)\rangle$. "Pure" realizations lead to superselection rules for quantum numbers l_i (cf. [5]) whereas "mixed" ones imply possibilities of critical ("threshold") phenomena and the spontaneous symmetry breaking (cf. [7]). And now we turn to some recent examples of explicit IDA applications focusing our attention only on key points.

3 G -invariant bi-photons and the $SU(2)$ -invariant treatment of unpolarized light

The first examples of applications of IDA to be examined deal with quantum-optical parametric models with m spatiotemporal and two polarization (\pm) light field modes whose Hamiltonians

$$H^2 = H_f + H_p^2, \quad H_f = \sum_{i=1}^m \omega_i \sum_{\alpha=+,-} a_{\alpha i}^+ a_{\alpha i}, \quad H_p^2 = \sum_{i=1}^m \sum_{\alpha,\beta=+,-} [g_{ij}^{\alpha\beta} a_{\alpha i}^+ a_{\beta j}^+ + g_{ij}^{\alpha\beta*} a_{\alpha i} a_{\beta j}] \quad (6)$$

are quadratic in field operators and c -numbers $g_{ij}^{\alpha\beta}$ determine concrete parametric processes [13, 20, 24]. Their simplest one-mode version ($m = 1, \alpha = +(-)$) has the invariance group $G_i^0 = C_2 = \{c_{k2} = \exp(i\pi k a^+ a), k = 0, 1\}$ acting on the Fock space $L_F(1) \equiv L_F(1; 0) = L(H^2)$ as follows: $a^+ \rightarrow c_{k2} a^+$. The dual pair is ($G_i = C_2, g^D = su(1, 1) = \text{Span}\{Y_0 = a^+ a/2 + 1/4, Y^+ = a^{+2}/2, Y = a^2/2\} \sim Sp(2, R)$), and the decomposition (3) is trivial: $L_F(1) = L(0) + L(1/2)$ where the eigenvalue $l_0 = \kappa/2$ of the operator $R_0 = a^+ a/2 - [a^+ a/2]$ ($[x]$ is the "entire part" of x), connected with the lowest weights k of the $su(1, 1)$ IRs realized on $L_F(1) : \kappa = 2k - 1/2$, determines the number $N_{up} = \kappa$ of un-paired photons in $L(l_0) = \text{Span}\{(Y^+)^{\mu}(a+)^{\kappa}|0\rangle\}$. The "particular" space $L(0)$ consists of bi-photons Y^+ and contains states $|\beta\rangle = \exp(\beta Y^+ - \beta^* Y)|0\rangle$ of the so-called "squeezed vacuum" light [30]. However, more interesting examples of CS in quantum optics due to symmetry have been found recently by using a specific polarization invariance of light fields.

Indeed, the free field Hamiltonian H_f in (6) is invariant with respect to the group $G_i^0 = \prod_{i=1}^m U^i(2) \subset Sp(4m, R)$ where $U^i(2) = \{\exp(i\gamma N_i + i\eta_0 P_0(i) + \eta_1 P_+(i) - \eta_1^* P_-(i))\}$, $N_i = \sum_{\alpha=+,-} N_{\alpha i}$ ($N_{\alpha i} = a_{\alpha i}^+ a_{\alpha i}$) is the photon number operator of the i -th spatiotemporal mode and $P_0(i) = [N_{+i} - N_{-i}]/2$, $P_{\pm}(i) = a_{\pm i}^+ a_{\mp i}$ are generators of the $SU(2)_p^i \subset U(2)^i$ subgroups defining the polarization $P(i)$ -quasispins (related to the polarization Stokes vector operators of single spatiotemporal modes) [24]. The group G_i^0 contains the $SU(2)_p$ subgroup generated by the total P -quasispin operators $P_\alpha = \sum_{i=1}^m P_\alpha(i)$ and enabled us to reveal hidden CS and to examine new collective phenomena connected with "polarization clusterizations" of light field modes [13, 24].

Really, the $SU(2)_p$ group acts on $L_F(2m) \equiv L_F(2m; 0)$ complementarily to the $so^*(2m)$ algebra generated by operators $E_{ij} \equiv \sum_{\alpha=\pm} a_{\alpha i}^+ a_{\alpha j} \in u(m)$ and $SU(2)_p$ -invariants $X_{ij}^+ = a_{+i}^+ a_{-j}^+ - a_{-i}^+ a_{+j}^+ : [P_\alpha, X_{ij}^+] = 0, \alpha = 0, +, -, X_{ij} = (X_{ij}^+)^+$. The decomposition (3) of $L(H) = L_F(2m)$ with respect to the dual pair ($G_i^p = SU(2)_p, g^D = so^*(2m)$) contains the infinite number of the

$SU(2)_p \otimes SO^*(2m)$ -invariant subspaces $L(l_0 = p) = L(p) = \text{Span}\{|p; \mu; \nu\rangle\}$ labeled by values p of the total P -quasispin which also determine the Casimir operator values of the $so^*(2m)$ IRs realized on $L_F(2m)$ and are measured in experiments with "polarization noises" [24, 13]. Basic vectors $|p; \mu; \nu \equiv [n_i, p_j]\rangle$, specified by the P_0 eigenvalue μ (helicity), photon numbers n_i and "intermediate" cluster quasispins p_j , have, in general, the form $|p; \mu; \nu\rangle = \mathcal{P}_\nu^{p; so^*(2m)}(X_{ij}^+)|p; \mu\rangle$ where the $so^*(2m)$ "pseudovacuum" vectors $|p; \mu\rangle = \mathcal{P}_\mu^{p; su(2)_p}(a_{\pm i}^+, Y_{ij}^+)|0\rangle$ are given by polynomials in $a_{\pm i}^+$ and P_0 -invariant operators $Y_{ij}^+ = (a_{+i}^+ a_{-j}^+ + a_{-i}^+ a_{+j}^+)/2 : [P_0, Y_{ij}^+] = 0$ which are direct analogs of Bogolubov's pairs in superfluidity. The operators $Y_{ij}^+, Y_{ij}^- = (Y_{ij}^+)^+$ extend the algebra $so^*(2m)$ to the algebra $u(m, m)$ acting on $L_F(2m)$ complementarily to the polarization subalgebra $u(1)_p = \text{Span}\{P_0\}$. From the physical point of view quantities X_{ij}^+, Y_{ij}^+ may be interpreted, respectively, as creation operators of P -scalar and P_0 -scalar bi-photon kinematic clusters determining, in fact, two classes of unpolarized light (UL) associated, respectively, with the "particular" subspaces $L(0) \equiv L(p=0)$ and $L'(0) \equiv L'(\mu=0) = \text{Span}\{|p; \mu=0; \nu\rangle\}$ [24].

Indeed, in [24] we proved that quantum states $|\rangle \in L(0), L'(0)$ satisfy the familiar definition of UL: $\mathcal{P} \propto [< P_0 >^2 + < P_1 >^2 + < P_2 >^2]^{1/2} = 0$ (\mathcal{P} is the light polarization degree, $P_\pm = (P_1 \pm iP_2)$, the symbol $< \dots >$ denotes both statistical and quantum averages) and, besides, extra (polarization "classicality" and "squeezing") conditions:

$$a) < |P_{i=1,2,0}^s| > = 0 \quad \forall s \geq 2, |\rangle \in L(0); \quad b) < |P_0^s| > = 0 \quad \forall s \geq 2, |\rangle \in L'(0) \quad (7)$$

States $|\rangle \in L(0), |\rangle \in L'(0)$ (P -and P_0 -scalar light in terminology [24]) are natural (and "particular" due to Eqs. (7)) representatives of two (introduced in [33] and named as P -and P_0 -invariant light in [25]) kinds of UL which obey general invariance conditions used in [34, 33] (in different forms) for more strong (in comparison with the above familiar) definitions of UL retaining some features of the natural (thermal) UL. Namely, states of P -invariant light satisfy the conditions

$$a) \text{Tr}[S\rho S^\dagger A(\{P_\alpha\})] = \text{Tr}[\rho A(\{P_\alpha\})] \equiv < A(\{P_\alpha\}) > \iff b) S\rho S^\dagger = \rho \quad (8)$$

for arbitrary P_α -dependent observables $A(\{P_\alpha\})$ or field density operators ρ (and appropriate quasiprobability functions) with any $S = \exp(ib_0 P_0 + b_1 P_+ - b_1^* P_-) \in SU(2)_p$ while states of P_0 -invariant light obey Eqs. (8) with $S = \exp(ib_0 P_0), \exp(i\pi P_2) \in SU(2)_p$. Emphasize, however, that P_0 -and P -scalar types of UL are due to strong phase correlations between photons unlike familiar states of UL generated by randomizing mechanisms. Note also that, in fact, the usual definition of arbitrary UL states ($\mathcal{P} = 0$) can be given in the form (8a) with any $S \in SU(2)_p$ if taking in it **only linear** functions $A(\{P_\alpha\})$ [25]. All these observations lead to a new treatment of (quantum and classical) UL states based on their $SU(2)_p$ invariance properties and to a natural division UL into two classes: 1) the weak UL having a characteristic property (8a) with any $S \in SU(2)_p$ only for first moments $< P_\alpha >$ (measured in standard polarization experiments) and 2) the strong UL possessing invariance properties (8) for higher moments and including P_0 -and P -invariant light.

So, taking into account only the $SU(2)_p$ invariance of H_f we have found in $L_F(2m)$ hidden kinematic CS ("polarization domains") described by subspaces $L(p)$ and $L'(\mu)$ which, according to general remarks of Section 2, can be realized "physically" with the help of G_i^p -invariant interaction Hamiltonians H_I of two kinds: 1) $H_I = H_I^{X,Y}$ depending only on bi-photon variables $Y_{ij}, Y_{ij}^+; X_{ij}, X_{ij}^+$ and G_i^p -scalar coupling constants; 2) $H_I = H_I^{cov}$ containing "free" photon operators $a_{\pm i}^+$ and G_i^p -covariant coupling parameters describing (phenomenologically) the chiral $SU(2)$ -symmetry of the matter (that, perhaps, is realized in some of biophysical models) [24]. The simplest examples of $H_I^{X,Y}$ are obtained from Eqs. (6) by imposing conditions: $g_{ij}^{+-} = \mp g_{ij}^{-+} = \tilde{g}_{ij}$ and $g_{ij}^{\alpha\beta} = 0$ otherwise in H_f ; actually, their (X_{ij}, X_{ij}^+) -independent versions were used for producing P_0 -scalar light (as states $\exp(\beta Y_{11}^+ - \beta^* Y_{11})|0\rangle$ of the "two-mode squeezed vacuum" [30]) while the problem of an experimental production of P -scalar light is not yet solved [24].

4 Coherent clusters in nonlinear models of quantum optics

The examples of applications of IDA using generalized Weyl-Howe dual pairs (G_i, \hat{g}) are yielded by generalizations of models (6) describing multiphoton scattering processes and quantum matter-radiation interactions [20]; their simplest versions are given by Hamiltonians

$$H_{mp} = \sum_{i=1}^m \omega_i a_i^+ a_i + \omega_0 a_0^+ a_0 + \sum_{1 \leq i_1 \leq i_2 \dots \leq i_n \leq m} [g_{i_1 \dots} (a_{i_1}^+ \dots a_{i_n}^+) a_0 + g_{i_1 \dots}^* (a_{i_1} \dots a_{i_n}) a_0^+], \quad n \geq 2 \quad (9)$$

where polarization labels are omitted in subscripts "i" and non-quadratic parts of H_{mp} describe, in particular, higher harmonics generation ($H_{mp} = H_{hg}$ when $m = 1$) and frequency conversions ($H_{mp} = H_{fc}$ when $m = n$ and only $g_{12\dots n} \neq 0$) whereas models of matter-radiation interactions are obtained via replacing in Eq. (9) the "pump" mode a_0^+ by "atomic" operators [20, 26].

The general Hamiltonians (9) have the invariance groups $G_i^{mp} = C_n = \{\exp(i2\pi ka^+ a/n), k = 0, 1, \dots, n-1\} \subset \prod_j U^j(1) = \exp(i\lambda_j a_j^+ a_j) = G_i^0$ whereas their specifications may have extra factors $\exp(i\beta_j R_j)$ related to dynamic constants (integrals of motion) $R_j \in \text{Span}\{N_i = a_i^+ a_i\}$ describing additional interaction symmetries; for instance, models H_{hg} have dynamic constants $R_1 = (N_1 + nN_0)/(1+n)$. Groups G_i^{mp} form on the Fock spaces $L_F(n+1)$ generalized dual pairs $(G_i^{mp}, g_{mp}^D = \hat{g}^Y(n+1))$ together with polynomial Lie algebras $\hat{g}^Y(n+1) = u(n+1) + y(n;1)$ obtained via the mapping (4) as extensions of the Lie algebras $u(n+1) = \text{Span}\{E_{ij} = a_i^+ a_j\}$ by coset spaces $y(n;1) = \text{Span}\{Y_{i_1 \dots i_n;0}^+ = a_{i_1}^+ \dots a_{i_n}^+ a_0, Y_{i_1 \dots i_n;0} = (Y_{i_1 \dots i_n;0}^+)^+\}$; herewith commutators $[Y_{i_1 \dots i_n;0}, Y_{j_1 \dots j_n;0}^+]$ are polynomials in E_{ij} [17]. Such an introduction of G_i^{mp} -invariant collective variables $E_{ij}, Y_{\dots;0}^+, Y_{\dots;0}$ enables us to rewrite Hamiltonians H_{mp} in the linear form (2) with respect to $E_{ij}, Y_{\dots;0}, Y_{\dots;0}^+$ and to apply the $\hat{g}^Y(n+1)$ formalism for revealing hidden CS and examining collective dynamic peculiarities in models (9) which slip off within standard studies [13, 27].

In order to elucidate basic ideas of such applications we restrict our analysis by models with Hamiltonians H_{hg} when $\hat{g}^Y(n+1)$ are reduced to the polynomial Lie algebras $su_{pd}(2) = \text{Span}\{Y_0 = (N_1 - N_0)/(1+n), Y_+ = (a_1^+)^n a_0, Y_- = (Y_+)^+\}$ with commutation relations

$$[Y_0, Y_\pm] = \pm Y_\pm, \quad [Y_-, Y_+] = \Phi(Y_0; R_1) \equiv \Psi(Y_0 + 1; R_1) - \Psi(Y_0; R_1), \quad [Y_\alpha, R_1] = 0 \quad (10)$$

resembling those for $su(2)$ but with polynomial structure functions $\Psi(Y_0; R_1) = (R_1 - Y_0 + 1)(nY_0 + R_1)^{(n)}(A^{(n)}) \equiv A(A-1) \dots (A-n+1)$ [26]; note that R_1 -dependence of $\Phi(Y_0; R_1)$, in fact, "intertwines" $G_i^{hg} = C_n \otimes \exp(i\beta R_1)$ and $g^D = su_{pd}(2)$ in an algebraic object resembling the semidirect product of groups (cf. [3, 5]). Then Hamiltonians H_{hg} are expressed by linear functions

$$H_{hg} = aY_0 + bY_+ + b^*Y_- + cR_1, \quad a = n\omega_1 - \omega_0, \quad b = g_{1\dots 1}, \quad c = (\omega_1 + \omega_0) \quad (11)$$

in the generators Y_α and dynamic constant R_1 , and the decomposition (3) of $L(H) = L_F(n+1)$ with respect to $(G_i^{hg}, su_{pd}(2))$ contains the infinite number of the $su_{pd}(2)$ -irreducible s -dimensional subspaces $L([l_i]) = \text{Span}\{(Y_+)^n |[l_0]\rangle, |[l_0]\rangle = (a_1^+)^{\kappa} (a_0^+)^s |0\rangle, \kappa = 0, \dots, n-1, s \geq 0\}$ labeled by eigenvalues $l_0 = (\kappa - s)/(1+n), l_1 = (\kappa + ns)/(1+n)$ of R_i where R_0 is determined from the identity: $\Psi(R_0; R_1) \equiv \Psi(Y_0; R_1) - Y_+ Y_-$ defining the $su_{pd}(2)$ Casimir operator [27].

This " $su_{pd}(2)$ -cluster" formulation of models entails a dimension reduction of physical tasks and an explicit "geometrization" of model dynamics manifesting already at the classical level of examination. So, e.g., the decomposition (3) implies the representation of model phase spaces C^{n+1} as fiber bundles: $C^{n+1} = \cup_{[l_i]} \mathcal{A}([l_i])$ where $su_{pd}(2)$ -invariant dynamic manifolds $\mathcal{A}([l_i])$ are Abelian varieties corresponding to spaces $L([l_i])$ and given (using the mean-field approximation) in dynamic variables $\bar{Y}_\alpha = \langle Y_\alpha \rangle$ as follows: $\mathcal{A}([l_i]) = \{\bar{Y}_\alpha : 2\bar{Y}_+ \bar{Y}_- = \Psi(\bar{Y}_0; l_1) + \Psi(\bar{Y}_0 + 1; l_1)\}$. Then, states belonging to a fixed manifold $\mathcal{A}([l_i])$ (or $\mathcal{A}([\bar{R}_i])$ in the general case) will evolve in it under action of Hamiltonian flows with Hamiltonian functions $\mathcal{H} = a\bar{Y}_0 + b\bar{Y}_+ + b^*\bar{Y}_- + c\bar{R}_1$. Herewith (approximate) dynamic trajectories are determined as intersections of manifolds $\mathcal{A}([\bar{R}_i])$ with energy planes $\mathcal{H} = E$ that enables us to determine some peculiarities of model dynamics [27].

These considerations become more transparent if using "quasi-spin" reformulations of the models (11) in terms of the $su(2)$ generators V_α connected with Y_α via the mapping (5): $V_0 = Y_0 - R_0 - J$, $V_+ = Y_+[\varphi(V_0)]^{1/2}$, $\varphi(V_0) = (J + V_0 + 1)(J - V_0)/\Psi(Y_0 + 1; R_1)$, $Y_- = (Y_+)^+$ (J is the $su(2)$ highest weight operator with eigenvalues $j = s/2$) [20]. Then the Hamiltonians (11) are represented by nonlinear functions

$$H = aV_0 + bV_+[\varphi(V_0)]^{-1/2} + b^*[\varphi(V_0)]^{-1/2}V_- + cR_1 + a(R_0 + J) \quad (12)$$

in the " $su(2)$ -cluster" variables V_α , and fiber bundle representations of phase spaces C^{n+1} contain $SU(2)$ -invariant "Bloch spheres" $S_j^2 : \bar{V}_0^2 + \bar{V}_1^2 + \bar{V}_2^2 = j^2$ instead of $su_{pd}(2)$ -invariant manifolds $\mathcal{A}([l_i])$ while energy planes are replaced by nonlinear energy surfaces $\langle H \rangle = E$. Furthermore, these "quasi-spin" reformulations enable us to get new (in comparison with obtained earlier) $su(2)$ -cluster quasiclassical solutions of spectral and evolution tasks using techniques of the $SU(2)$ coherent states $|\phi_0; \alpha\rangle = S_V(\alpha)|\phi_0\rangle \in L(H)$, $S_V(\alpha) = \exp(\alpha V_+ - \alpha^* V_-)$ which can be of "spin-like" type (when $|\phi_0\rangle \in L([l_i])$) or $su(2)$ -reducible (when $|\phi_0\rangle \in L(H)$) [27].

For example, energy eigenstates $|E([l_i]; v)\rangle$ and spectra $\{E([l_i]; v)\}$ can be approximated by means of standard variational schemes with using $SU(2)$ coherent states $S_V(\xi)\mathcal{N}([l_i]; v)V_+^v|[l_i]\rangle = |[l_i]; v; \xi\rangle$ as trial functions. Namely, we find approximate eigenstates $|E^{qc}([l_i]; v)\rangle = |[l_i]; v; \xi\rangle$ and eigenenergies $E^{qc}([l_i]; v) = \langle [l_i]; v; \xi | H | [l_i]; v; \xi \rangle$ where values of the parameter $\xi = r \exp(-i\theta)$ are determined by the stationarity conditions

$$\frac{\partial \mathcal{H}([l_i]; v; \xi)}{\partial \theta} = 0, \quad \frac{\partial \mathcal{H}([l_i]; v; \xi)}{\partial r} = 0, \quad \mathcal{H}([l_i]; v; \xi) = \langle [l_i]; v; \xi | H | [l_i]; v; \xi \rangle \quad (13)$$

for the energy functional $\mathcal{H}([l_i]; v; \xi)$. In fact, in such a way we get $\exp(-i\theta) = b/|b|$ and a whole series of competitive potential solutions for values r ; their final selection may be made with the help of a "quality criterion" using the "energy error" functionals introduced in [20]. Similarly, an appropriate quasiclassical dynamics is described by the classical Hamiltonian equations [27]

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}, \quad \mathcal{H} = \langle \phi_0; z(t) | H | [\phi_0; z(t)] \rangle, \quad q = \theta, \quad p \equiv \langle \phi_0; z(t) | Y_0 | \phi_0; z(t) \rangle \quad (14)$$

for "motion" of the canonical parameters p, q of the $SU(2)$ coherent states $|\phi_0; z(t)\rangle = S_V(z(t))|\phi_0\rangle$ ($z = -r \exp(i\theta)$) as trial functions in the time-dependent Hartree-Fock variational scheme. Note that solutions of Eqs.(13)-(14) smoothly approximate exact ones and catch explicitly quantum cooperative features of models at the quasiclassical levels [27].

5 Generalized dual pairs in the theory of composite fields

Another area of a "natural" appearance of generalized dual pairs $(G_i, g^{DS} = \hat{g})$ is the algebraic analysis [13, 23] of composite fields with internal (gauge) symmetries [5] which generalizes basic ideas of the paraquantization [8, 35] and implements in a sense the method of fusion by L. de Broglie [36]. Actually, the simplest example of such an analysis (but without introducing dual pairs and non-linear Lie algebras \hat{g}) was given in [31] by means of using n -boson one-mode versions

$$H^n = \omega_1 a_1^+ a_1 + g Y_{1\dots}^+ + g^* Y_{1\dots}, \quad Y_{1\dots}^+ = a_1^+ \dots a_1^+ = (a_1^+)^n \quad (15)$$

of Hamiltonians (6) to describe resonance states in particle physics; later it was generalized on multimode cases to study multiphoton processes in quantum optics (see [13] and references therein).

Specifically, in [31] it was shown that operators $Y^+ \equiv Y_{1\dots}^+$ describe n -particle kinematic clusters which display unusual (para)statistics and correspond to generalized asymptotically free fields realized on the Fock space $L_F(1)$. In fact, the operators $Y^+, Y = (Y^+)^+, Y_0 = a_1^+ a_1/n \equiv E_{11}/n$ satisfy [13] (non-canonical) commutation relations (10) of the $su_{pd}(1, 1)$ algebra with the structure polynomial $\Psi(Y_0) = (E_{11})^{(n)}$ and, besides, extra multi-linear relations: $ad_Y^{n+1} Y^+ = ad_Y^n (ad_Y Y^+) =$

$0, ad_Y Y^+ \equiv [Y, Y^+]$, generalizing (for $n \geq 3$) trilinear parastatistical Green's relations [8, 35]. Thus, we get an action of the generalized dual pair ($G_i = C_n = \{\exp(i2\pi k a_1^+ a_1)/n\}, \hat{g} = su_{pd}(1, 1)$) on the space $L_F(1)$. The appropriate decomposition (3) contains the subspaces $L([l_0 = \kappa/n]) = Span\{(Y^+)^{\eta}[l_0]\}, |[l_0]\rangle = (a+)^{\kappa}|0\rangle\}$, $\kappa = 0, 1, \dots, n-1$, describing coherent mixtures of constant numbers κ of uncoupled bosons a_1^+ and of varying in time numbers N_Y of Y -clusters. However, operators N_Y have not standard (for (papr)fields) bilinear in Y, Y^+ forms [8] but they can be expressed (due to the evident identity $\Psi(Y_0) = Y^+ Y$ on $L_F(1)$) as nonlinear functions in the bilineals $Y^+ Y, Y Y^+$ [13]: $N_Y = (E_{11} - nR_0)/n = [a_1^+ a_1/n] = [E_{11}/n], E_{11} = \varphi(Y^+ Y) \equiv n\Psi^{-1}(Y_0)$ as it is the case for algebras $A(K)$ describing non-standard statistics [35]. Therefore, at best the quantities Y^+, Y can be set in correspondence only to parafield (when $n = 2$) quanta [8, 35] rather than to certain asymptotically free particles [13]. Nevertheless, one can construct from them operators $W^+ = W^+(\{Y_i\}), W = (W^+)^+$ obeying canonical commutation relations $[W, W^+] = 1$, having the standard number operators $N_W = W^+ W (= N_Y)$ and corresponding to quanta of asymptotically free multi-boson fields (which can be realized in subspaces $L([0])$ in "pure forms"). Actually, two equivalent forms [13, 31, 20]:

$$W^+ = Y^+ \sum_{r \geq 0} c_r (Y^+)^r (Y)^r = Y^+ [(Y_0 - R_0 + 1)/(E_{11} + n)^{(n)})]^{1/2}, \quad W = (W^+)^+ \quad (16)$$

were found for such W^+, W where the second one is, a specification of the mapping (5).

The analysis above has been generalized [17] by means of: 1) using " m "-mode extensions of models (15) with C_n -invariant interaction Hamiltonians $\sum_{1 \leq i_1 \dots \leq m} [g_{i_1 \dots} Y_{i_1 \dots}^+ + g_{i_1 \dots}^* Y_{i_1 \dots}]$, $Y_{i_1 \dots}^+ = a_{i_1}^+ \dots a_{i_n}^+$; 2) considering their analogs with non-Abelian groups $G_i = SU(n)$ (whose Hamiltonians are obtained by the substitutions: $a_i^+ a_i \rightarrow (\mathbf{a}_i^+ \cdot \mathbf{a}_i) \equiv \sum_{j=1}^n a_{ji}^+ a_{ji}$, $Y_{i_1 \dots}^+ \rightarrow X_{i_1 \dots}^+ \equiv \sum_{(jk)} \epsilon_{j_1 \dots j_n} a_{j_1 i_1}^+ \dots a_{j_n i_n}^+$ ($\epsilon_{j_1 \dots j_n}$ is the totally antisymmetric tensor)); 3) involving both boson and fermion variables. These procedures yield a variety of generalized dual pairs; for instance, when using two first ones we get dual pairs $(C_n, osc^Y(m; (n)))$ and $(SU(n), osc^X(m; 1^n))$ where $osc^Y(m; (n))$ and $osc^X(m; 1^n)$ are extensions of the unitary algebras $u(m) = Span\{E_{ij}, E_{ij} = a_i^+ a_j\}$ and $E_{ij} = (\mathbf{a}_i^+ \cdot \mathbf{a}_j)\}$ by their symmetric ($Y_{i_1 \dots}^+, Y_{i_1 \dots}$) and skew-symmetric ($X_{i_1 \dots}^+, X_{i_1 \dots}$) tensor operators [17, 23]. The operators $X_{i_1 \dots}^+, X_{i_1 \dots}$ and $Y_{i_1 \dots}^+, Y_{i_1 \dots}$ satisfy non-canonical commutation relations whose right sides depend on E_{ij} (and on the $SU(n)$ Casimir operators for $osc^X(m; 1^n)$) and obey (due to the invariant theory [1, 29]) certain extra "bootstrap" relations ("syzygies") of the type: $Y_{1 \dots 1} Y_{2 \dots 2} = Y_{21 \dots 1} Y_{12 \dots 2}$ [17, 13] which are similar to those occurring in quantum field theories with constraints [5, 8] and in non-standard quantization schemes discussed in [35]. All this entails unusual statistical and other features of G_i -invariant clusters associated with $X_{i_1 \dots}^+, Y_{i_1 \dots}^+$ and complicates extensions of the one-mode analysis above [17]. Specifically, the task of obtaining m -mode generalizations

$$W_a^+ = \sum_{i_1 \dots i_n} (Y_{i_1 \dots i_n}^+ / X_{i_1 \dots i_n}^+) f_{i_1 \dots i_n}^a (\{E_{ij}\}), \quad [W_a, W_b^+] = \delta_{ab}, \quad W_a = (W_a^+)^+ \quad (17)$$

of the mapping (16) is, in general, fairly difficult owing to "syzygies" between Y/X -clusters (and resembles the "reducibility problem" for algebras $A(K)$ [35]).

When determining explicit expressions for $f_{\dots}(\dots)$ in Eqs. (17) (and in their generalizations, e.g., for constructing $W_a^+ \in A(K)$) we get an effective tool for analyzing composite field models with internal G_i -symmetries at the algebraic and quasi-particle levels (including a new insight into some "old problems", such as, e.g., the quark confinement [3, 10]). Furthermore, examining the limit " $m \rightarrow \infty$ " and involving spatiotemporal variables and symmetries into consideration, one can also construct in terms of "quanta" W_a appropriate "physical" (asymptotically free) composite fields [13] and, then, develop for them standard theories including non-linear (due to Hamiltonian forms) evolution equations and their soliton/instanton solutions [7]; herewith discrete quantum numbers l_i labeling subspaces $L([l_i])$ in (3) may display themselves as specific topological charges. In particular, in such a way, using suitable analogs of Eq. (16) for P/P_0 -scalar biphotos [24], we

answer in the affirmative within quantum optics the problem of existence of UL waves put by A. Fresnel in the beginning of XIX century and having the negative solution within the framework of classical electrodynamics due to the vector nature of the Maxwell equations [4, 6].

6 Conclusion

So, we formulated mathematical grounds of IDA and showed its physical meaning "in action". In conclusion we briefly discuss some ways of applying and developing results obtained.

The general constructions of Sections 2,5 may be applied for the systematic search of hidden CS within different areas of quantum many-body physics by using known dual pairs [28, 20] and for developing field theories with "hidden quantum variables" and unusual statistics [8, 13, 35] (including the problem of consistency of the Poincare symmetry with dynamic ones [7, 11, 13]). On other hand, they are useful in solving appropriate "inverse problems" [16]: to display hidden symmetries G_i and "pre-particles" from analyzing spectroscopic data for complex systems associated with IRs of certain dynamic algebras g^D (that is of very importance when interaction Hamiltonians are determined phenomenologically). For this aim it is worth-while to enlarge lists of dual pairs used by involving new classes of groups G_i and q-deformed oscillators into consideration [23].

More concrete results of Sections 3,4, firstly, can be used as general patterns of applying IDA in G_i -invariant many-body models and, secondly, open new lines of investigations in quantum optics. So, e.g., the above $SU(2)_p$ -invariant treatment of UL stimulates experiments on producing new states of quantum UL (especially, of P -scalar light), studies of interactions of these states with material media [13] and their applications in communication theory, spectroscopy of anisotropic media and biophysics [24]. At the same time "quasi-spin" formulations and $su(2)$ -cluster quasiclassical approximations in models (9) outline (related to geometric quantization schemes [32]) ways of "geometrization" of dynamics in models of strongly interacting subsystems and, simultaneously, can be used to reveal new collective phenomena in such models, including topological features of Hamiltonian flows determined by Eqs. (14) at the different quasiclassical levels [27].

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